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Random wetting transition on the Cayley tree: a disordered first-order transition with two correlation length exponents

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Abstract

We consider the random wetting transition on the Cayley tree, i.e. the problem of a directed polymer on the Cayley tree in the presence of random energies along the left-most bonds. In the pure case, there exists a first-order transition between a localized phase and a delocalized phase, with a correlation length exponent $\nu_{\text{pure}} = 1$. In the disordered case, we find that the transition remains first order, but that there exists two diverging length scales in the critical region: the typical correlation length diverges with the exponent $\nu_{\text{typ}} = 1$, whereas the averaged correlation length diverges with the bigger exponent $\nu_{\text{av}} = 2$ and governs the finite-size scaling properties. We describe the relations with previously studied models that are governed by the same ‘infinite disorder fixed point’. For the present model, where the order parameter is the contact density $\theta_L = l_a/L$ (defined as the ratio of the number l_a of contacts over the total length L), the notion of ‘infinite disorder fixed point’ means that the thermal fluctuations of θ_L within a given sample become negligible on a large scale with respect to sample-to-sample fluctuations. We characterize the statistics over the samples of the free energy and of the contact density. In particular, exactly at criticality, we obtain that the contact density is not self-averaging but remains distributed over the samples in the thermodynamic limit, with the distribution $\mathcal{P}_{T_c}(\theta) = 1/(\pi\sqrt{\theta(1-\theta)})$.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In pure phase transitions, the approach to criticality is usually governed by a single correlation length exponent ν that describes all finite-size scaling properties. In the presence of frozen disorder, however, the distribution of correlation functions may become very broad and disorder-averaged values can become atypical i.e. dominated by rare events. This phenomenon was first studied in one-dimensional classical spin systems [1–3], where correlation functions

can be expressed as the product of random numbers, and have then been found in higher-dimensional systems, such as the 2D random q -state Potts model [5] and the random transverse field Ising chain [4] (which is equivalent to the classical 2D McCoy–Wu model). It is important to stress that both the typical and averaged correlations are actually important, depending on the physical quantities one wants to study [1, 2, 4]. It turns out that close to a phase transition, the typical correlation length ξ_{typ} and the disorder-averaged correlation length ξ_{av} may have different critical behaviors. The best understood example is the random transverse field Ising chain, which has been studied in more detail by D S Fisher via a strong disorder renormalization approach [4] to obtain $\nu_{\text{typ}} = 1$ and $\nu_{av} = 2$. These two exponents $\nu_{\text{typ}} = 1$ and $\nu_{av} = 2$ also occur in other disordered models that are described by the same ‘infinite disorder fixed point’ (see the review [6] and references therein). In this paper, we still present another realization of this ‘infinite disorder fixed point’ with $\nu_{\text{typ}} = 1$ and $\nu_{av} = 2$, as a random wetting transition on the Cayley tree.

Our physical motivation to consider such a model of random wetting on the Cayley tree was to better understand the similarities and differences with two other types of models involving directed polymers and frozen disorder:

- (i) For the problem of the directed polymer in a random medium on the Cayley tree [7], there exists a freezing transition toward a low-temperature phase of finite entropy, where the polymer is essentially frozen along the random optimal path. Finite-size properties in the critical region [8, 9] have revealed the presence of two distinct correlation length exponents $\nu = 2$ and $\nu' = 1$. In the random wetting model considered in this paper, the difference is that the random energies are not on all bonds of the Cayley tree, but only on the bonds of the left-most path, so that the phase transition corresponds to a freezing along this boundary path.
- (ii) For the wetting [10] and the Poland–Scheraga model of DNA denaturation [11] with a loop exponent $c > 2$, where the corresponding pure transition is of first order, we have found numerically [12, 13] that in the presence of frozen disorder, the transition remains first order, but that two correlation length exponents $\nu_{\text{typ}} = 1$ and $\nu = 2$ appear. Since these results have been debated [14, 15], it is interesting to obtain analytically exactly the same critical properties in a closely related model. The precise relation between the two models is that the random wetting model considered in this paper corresponds to the limit of loop exponent $c \rightarrow \infty$ (since loops do not exist on the Cayley tree) of the model studied in [12] with the boundary conditions bound–unbound (see more details on the model in section 2.1 of [12]).

This paper is organized as follows. In section 2, we introduce the model and the interesting observables. In section 3, we discuss the statistical properties of the free energy and conclude that the wetting transition remains first order with a jump of the energy density. In section 4, we study the statistics of the attached and detached lengths, and give the consequences for the statistics of the order parameter in the critical region. We summarize our conclusions in section 6. Appendix A contains for comparison the analysis of the properties of the wetting transition in the pure case. In appendix B, we discuss the transition temperatures of the moments of the partition function in the disordered case.

2. Model and observables

2.1. Definition of the model

We consider a directed polymer of length L on a Cayley tree of branching ratio K . The total number of directed walks is simply

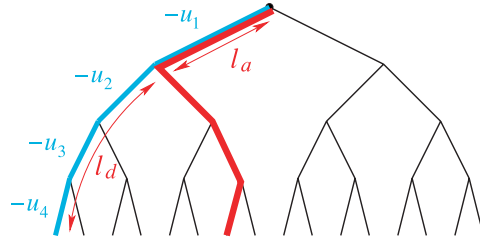


Figure 1. In this paper, we consider a Cayley tree of branching ratio K and a number L of generations ($K = 2$ and $L = 4$ in the figure), and we study the statistical physics of a directed polymer starting at the root in the presence of random energies $(-u_1, -u_2, \dots, -u_L)$ along the left-most bonds. A configuration of the directed polymer is characterized by an attached length $0 \leq l_a \leq L$ along these left-most bonds and a detached length $l_d = L - l_a$ ($l_a = 1$ and $l_d = 3$ in the figure). The delocalized phase corresponds to a finite l_a as $L \rightarrow +\infty$, whereas the localized phase corresponds to a finite l_d as $L \rightarrow +\infty$. The order parameter is the contact density $\theta = l_a/L$.

$$Y_L = K^L. \tag{1}$$

The corresponding delocalized free energy reads

$$F_L^{\text{deloc}} \equiv -T \ln Y_L = -TL \ln K \tag{2}$$

and corresponds to an entropy of $(\ln K)$ per bond.

We now consider the random wetting model, where the left-most bonds are characterized by independent random energies $(-u_i)$ with $i = 1, 2, \dots, L$ (see figure 1). As an example, they can be drawn from the Gaussian distribution of average value $u_0 > 0$ and variance Δ^2 :

$$p(u) = \frac{1}{\sqrt{2\pi} \Delta} e^{-\frac{(u-u_0)^2}{2\Delta^2}}. \tag{3}$$

2.2. Recurrence satisfied by the partition function

To write a recurrence for the partition function, it is convenient to label the random energies from the bottom, instead of labeling from the root (shown in figure 1):

$$\tilde{u}_i = u_{L-i}. \tag{4}$$

The partition function Z_L then satisfies the simple recurrence

$$Z_{L+1} = e^{\beta \tilde{u}_{L+1}} Z_L + (K - 1)Y_L \tag{5}$$

with the initial condition $Z_0 = 1$. It is convenient to introduce the ratio with respect to the free partition function of equation (1):

$$R_L \equiv \frac{Z_L}{Y_L} = \frac{Z_L}{K^L}. \tag{6}$$

The recurrence then becomes

$$R_{L+1} = \frac{e^{\beta \tilde{u}_{L+1}}}{K} R_L + \frac{(K - 1)}{K} \tag{7}$$

with the initial condition $R_0 = 1$.

2.3. Free-energy difference with the delocalized state

The difference between the free energy $F_L = -T \ln Z_L$ and the delocalized free energy of equation (2) is given by

$$F_L^{\text{diff}} \equiv F_L - F_L^{\text{deloc}} = -T \ln Z_L + T \ln Y_L = -T \ln R_L. \quad (8)$$

The excess free energy per monomer due to the wall in the thermodynamic limit $L \rightarrow +\infty$

$$f_\infty \equiv \lim_{L \rightarrow +\infty} \frac{F_L - F_L^{\text{deloc}}}{L} = -T \lim_{L \rightarrow +\infty} \frac{\ln R_L}{L} \quad (9)$$

characterizes the wetting transition: it vanishes in the delocalized phase $f_\infty(T > T_c) = 0$ and remains finite in the localized phase $f_\infty(T < T_c) < 0$.

2.4. Probability distributions of the attached length l_a and of the detached length l_d

Since there is no loop on the Cayley tree, the directed polymer cannot return to the wall after leaving it. A configuration can be thus decomposed into a length l_a attached to the wall starting at the root and a length $l_d = L - l_a$ detached from the wall (see figure 1). The thermal probability $Q_L(l_a)$ to have exactly l_a links attached to the wall in a given disordered sample reads

$$Q_L(l_a) = \frac{e^{\beta \sum_{i=1}^{l_a} u_i} [\delta_{l_a, L} + (K-1)K^{L-l_a-1} \theta(l_a < L)]}{Z_L}. \quad (10)$$

The numerator is the contribution to the partition function of the configurations that leave the wall after exactly l_a steps. The denominator is the partition function

$$Z_L = \sum_{l_a=0}^L e^{\beta \sum_{i=1}^{l_a} u_i} [\delta_{l_a, L} + (K-1)K^{L-l_a-1} \theta(l_a < L)] \quad (11)$$

so that the normalization of the probability distribution $Q_L(l_a)$ reads

$$\sum_{l_a=0}^L Q_L(l_a) = 1. \quad (12)$$

It will be convenient to also introduce the probability distribution of the detached length $l_d = L - l_a$:

$$P_L(l_d) \equiv Q_L(l_a = L - l_d) = \frac{e^{\beta \sum_{i=1}^{L-l_d} u_i} [\delta_{l_d, 0} + (K-1)K^{l_d-1} \theta(l_d \geq 1)]}{Z_L}. \quad (13)$$

2.5. Order parameter: contact density

A convenient order parameter of the transition is the contact density, which is directly related to the attached length l_a introduced in equation (10):

$$\theta_L \equiv \frac{l_a}{L}. \quad (14)$$

In particular, its thermal average in a given disordered sample is determined by the first moment of the distribution $Q_L(l_a)$ of equation (10):

$$\langle \theta_L \rangle = \frac{\langle l_a \rangle}{L} = \frac{1}{L} \sum_{l_a=0}^L l_a Q_L(l_a). \quad (15)$$

In the thermodynamic limit $L \rightarrow \infty$, it is expected to remain finite in the localized phase and to vanish in the delocalized phase.

2.6. Internal energy

The internal energy

$$E_L = -\partial_\beta \ln Z_L = -\partial_\beta \ln R_L \quad (16)$$

satisfies the recursion (see equation (7) for the corresponding recursion for the reduced partition function R_L)

$$E_{L+1} = \frac{e^{\beta \tilde{u}_{L+1}} R_L (-\tilde{u}_{L+1} + E_L)}{e^{\beta \tilde{u}_{L+1}} R_L + (K - 1)} \quad (17)$$

and the initial condition $E_0 = 0$. In terms of the attached length l_a introduced above (see equation (10)), it can also be written as

$$E_L = \frac{\sum_{l_a=0}^L \left(\sum_{i=1}^{l_a} u_i \right) e^{\beta \sum_{i=1}^{l_a} u_i} [\delta_{l_a, L} + (K - 1) K^{L-l_a-1} \theta(l_a < L)]}{Z_L}. \quad (18)$$

The energy per monomer in the thermodynamic limit $L \rightarrow +\infty$

$$e_\infty \equiv \lim_{L \rightarrow +\infty} \frac{E_L}{L} \quad (19)$$

remains finite in the localized phase $e_\infty(T < T_c) > 0$ and vanishes in the delocalized phase $e_\infty(T > T_c) = 0$. Its value can be obtained via the usual thermodynamic relation from the excess free energy f_∞ introduced in equation (9):

$$e_\infty = \partial_\beta (\beta f_\infty). \quad (20)$$

3. Statistical properties of the free energy

In this section, we discuss the statistics of the free-energy difference of equation (8) which depends only on the variable R_L introduced in (6). We first explain that R_L is a Kesten random variable and describe its statistical properties.

3.1. Analysis of the variable R_L as a Kesten random variable

The recursion of equation (7) takes the form of the recurrence of the Kesten variable [16],

$$R_{L+1} = a_{L+1} R_L + b, \quad (21)$$

where $b = (K - 1)/K$ is a constant and where

$$a_i = \frac{e^{\beta u_i}}{K} \quad (22)$$

are independent identically distributed random variables. The specific structure of a Kesten variable R_L consists in a sum of products of random variables:

$$R_L = \prod_{i=1}^L a_i + b \left(1 + \sum_{j=2}^L \prod_{i=j}^L a_i \right). \quad (23)$$

This discrete form has for a continuous analog the exponential functional,

$$\mathcal{R}_L = \int_0^L dx e^{\int_x^L dy F(y)}, \quad (24)$$

where $\{F(x)\}$ is the random process corresponding to the random variables $(\ln a_i)$ in the continuous limit.

This type of random variables, in either the discrete or continuous forms, appear in a variety of disordered systems, in particular in random walks in random media [17–24, 26, 27], in the classical random field Ising chain [28, 29] and in the quantum random transverse field Ising chain [30–32]. Thus, many properties have already been studied in detail in these previous works. In the following, we cite and translate these known results for the present context.

To describe the universal results near the critical point for large samples, there are only two relevant parameters:

- (i) The first important parameter is the averaged value

$$F_0 \equiv \overline{\ln a_i} = \beta \bar{u} - \ln K. \quad (25)$$

The critical point corresponds to the vanishing condition $\overline{\ln a_i} = 0$, so that the critical temperature of the present random wetting model reads

$$T_c = \frac{\bar{u}}{\ln K}. \quad (26)$$

(In particular, for the Gaussian distribution of equation (3), the critical temperature is $T_c = u_0/(\ln K)$ and is independent of variance Δ^2 .)

- (ii) The second important parameter is the variance of $(\ln a_i)$ that will be denoted by

$$2\sigma \equiv \overline{(\ln a_i)^2} - (\overline{\ln a_i})^2 = \beta(\overline{u_i^2} - (\bar{u}_i)^2) \quad (27)$$

to keep the same notations as in [24, 27, 32]. In particular, from the central theorem, the sum $\Sigma_N = \sum_{i=1}^N (\ln a_i)$ is then distributed asymptotically for large N with the Gaussian distribution:

$$P_N(\Sigma_N) \underset{N \rightarrow \infty}{\simeq} \frac{1}{\sqrt{4\pi\sigma N}} e^{-\frac{(\Sigma_N - NF_0)^2}{4\sigma N}}. \quad (28)$$

3.2. Delocalized phase $T > T_c$

For $T > T_c$, the parameter of equation (25) is negative $F_0 < 0$ and the random variable R_L remains a finite random variable as $L \rightarrow \infty$. Its probability distribution $P_\infty(R_\infty)$ is known to display the power-law decay [16, 28, 29],

$$P_\infty(R_\infty) \underset{R_\infty \rightarrow \infty}{\propto} \frac{1}{R_\infty^{1+\mu(T)}}, \quad (29)$$

where the exponent $\mu(T) > 0$ is determined by the condition

$$1 = \overline{a_i^\mu} = \frac{e^{\mu\beta u_i}}{K^\mu}. \quad (30)$$

In conclusion, the difference of equation (8) between the free energy F_L and the delocalized free energy F_L^{deloc} remains a finite random variable as $L \rightarrow \infty$

$$F_L^{\text{diff}} \equiv F_L - F_L^{\text{deloc}} \underset{L \rightarrow \infty}{\simeq} F^{\text{diff}} \equiv -T \ln R_\infty, \quad (31)$$

and its distribution presents the following exponential tail for $F_{\text{diff}} \rightarrow -\infty$ (see equation (29)):

$$P_{\text{deloc}}(F^{\text{diff}}) \underset{F_{\text{diff}} \rightarrow -\infty}{\simeq} e^{\beta\mu(T)F_{\text{diff}}}. \quad (32)$$

Example: for the case of the Gaussian distribution of equation (3), the condition of equation (30) becomes

$$1 = \frac{e^{\mu\beta u_0 + \frac{\mu^2\beta^2\Delta^2}{2}}}{K^\mu} \quad (33)$$

and the exponent μ reads

$$\mu^{\text{Gauss}}(T) = \frac{2T(T \ln K - \bar{u}_L)}{\Delta^2} = \frac{2(\ln K)T(T - T_c)}{\Delta^2}. \quad (34)$$

3.3. Critical point $T = T_c$

Exactly at criticality $T = T_c$, the parameter of equation (25) vanishes $F_0 = 0$. It is then known (see for instance [24, 25, 32] and references therein) that the leading behavior is of order

$$\ln R_L(T_c) \underset{L \rightarrow \infty}{\simeq} 2(\sigma L)^{1/2} w_c, \quad (35)$$

where w_c is a positive random variable of order $O(1)$. Moreover, it can be shown (see [32] and references therein) that the distribution of w_c is the half-Gaussian distribution (cf equations (24) and (32) in [32]):

$$P(w_c) \simeq \theta(w_c \geq 0) \frac{2}{\sqrt{\pi}} e^{-w_c^2}. \quad (36)$$

3.4. Localized phase $T < T_c$

For $T < T_c$, the parameter of equation (25) is positive $F_0 > 0$ and the random variable R_L will grow exponentially in L

$$\ln R_L(T < T_c) \underset{L \rightarrow \infty}{\simeq} LF_0 + 2\sqrt{\sigma L} w. \quad (37)$$

The random variable w of order $O(1)$ is distributed with the Gaussian distribution (cf equations (24) and (31) in [32]):

$$P(w) \simeq \frac{1}{\sqrt{\pi}} e^{-w^2}. \quad (38)$$

3.5. Summary of the critical behavior of the free energy and energy

In conclusion, the difference of equation (8) between the free energy F_L and the delocalized free energy F_L^{deloc} displays the following critical behavior:

$$\begin{aligned} F_L^{\text{diff}}(T > T_c) &= F_{\text{diff}} \quad \text{of order } O(1) \\ F_L^{\text{diff}}(T_c) &= -T_c 2(\sigma L)^{1/2} w_c \quad \text{with } w_c > 0 \quad \text{of order } O(1) \\ F_L^{\text{diff}}(T < T_c) &= -L(\bar{u} - T \ln K) - T 2\sqrt{\sigma L} w \quad \text{with } w \quad \text{of order } O(1), \end{aligned} \quad (39)$$

where the distributions of the random variables F_{diff} , w_c and w have been given in equations (32), (36) and (38). In the thermodynamic limit $L \rightarrow +\infty$, the excess free energy per monomer due to the wall defined in equation (9) is thus given by

$$f_\infty(T \geq T_c) = 0 \quad (40)$$

$$f_\infty(T \leq T_c) = -(\bar{u} - T \ln K) = -(T_c - T) \ln K. \quad (41)$$

The thermodynamic relation of equation (20) then gives the behavior of the energy per monomer:

$$e_\infty(T > T_c) = 0 \quad (42)$$

$$e_\infty(T < T_c) = -T_c \ln K = -\bar{u}. \quad (43)$$

The transition is thus first order, with a jump of the energy per monomer.

4. Statistical properties of the attached and detached lengths

In this section, we discuss the statistical properties of the attached length l_a or of the complementary detached length l_d whose distributions have been introduced in equations (10) and (13), respectively. This allows us to analyze also the statistics of the contact density of equation (14), which represents the order parameter, and of the internal energy of equation (18).

4.1. Typical behavior of $Q_L(l_a)$ in the delocalized phase

In the delocalized phase, the attached length l_a will remain a finite random variable as $L \rightarrow \infty$. The typical decay of $Q_L(l_a)$ is then given by (equation (10))

$$\overline{\ln Q_L(l_a)} = \beta l_a \bar{u} + (L - l_a) \ln K + \ln \frac{K-1}{K} - \overline{\ln Z_L} \quad (44)$$

$$\simeq -l_a (\ln K - \beta \bar{u}) + \ln \frac{K-1}{K} + (L \ln K - \overline{\ln Z_L}), \quad (45)$$

i.e. the decay is governed by the typical correlation length

$$\frac{1}{\xi_{\text{typ}}} = \ln K - \beta \bar{u} \quad (46)$$

that diverges with the correlation length exponent

$$\nu_{\text{typ}} = 1 \quad (47)$$

as in the pure case ($\nu_{\text{pur}} = 1$ of equation (A.8)).

4.2. Typical decay of $P_L(l_d)$ in the localized phase

In the localized phase, the detached length l_d will remain a finite random variable as $L \rightarrow +\infty$. The typical decay of $P_L(l_d)$ is then given by

$$\overline{\ln P_L(l_d)} = \beta(L - l_d)\bar{u} + l_d \ln K + \ln \frac{K-1}{K} - \overline{\ln Z_L} \quad (48)$$

$$\simeq -l_d(\beta \bar{u} - \ln K) + \ln \frac{K-1}{K} + (L\beta \bar{u} - \overline{\ln Z_L}), \quad (49)$$

i.e. the decay is governed by the typical correlation length

$$\frac{1}{\xi_{\text{typ}}} = \beta \bar{u} - \ln K \quad (50)$$

that diverges with the correlation length exponent

$$\nu_{\text{typ}} = 1 \quad (51)$$

as in the pure case ($\nu_{\text{pur}} = 1$ of equation (A.17)).

4.3. Statistics over the samples of the probability $q_L = Q_L(l_a = 0)$ of zero contacts

To understand why typical and disordered averaged behaviors can be different for the distributions $Q_L(l_a)$ and $P_L(l_d)$, it is convenient to consider first the particular case of the probability of zero contacts

$$q_L \equiv Q_L(l_a = 0) = P_L(l_d = L) = \frac{(K-1)K^{L-1}}{Z_L} = \frac{(K-1)}{K} \frac{1}{R_L} \quad (52)$$

because it depends only on the Kesten variable R_L whose statistics has been discussed in detail in section 3. We also note that q_L coincides with the expression of the flux J_L in the Sinai model, and we refer to [24] where many exact results have been derived for its probability distribution (see in particular equation (6.10) in [24]).

In the following, we concentrate on the critical region. From the statistics of R_L described in section 3, one has in particular that

- (i) in the delocalized phase, q_L remains a finite random variable as $L \rightarrow +\infty$
- (ii) in the localized phase, the probability q_L becomes exponentially small in L ,
- (iii) exactly at criticality, the variable q_L behaves as $q_L \sim e^{-w_c 2\sqrt{\sigma L}}$ where w_c is a positive random variable of order $O(1)$ distributed with the half-Gaussian distribution of equation (36). As discussed in [32], an important consequence of the critical statistics of equation (36) is that the averaged value is dominated by the rare samples having an anomalously small value of the scaling variable $0 \leq w \leq 1/\sqrt{L}$, and since the probability distribution $P(w_c)$ is finite at $w_c = 0$, one obtains the slow power-law decay

$$\overline{q_L(T_c)} \underset{L \rightarrow \infty}{\propto} \frac{1}{\sqrt{L}}, \tag{53}$$

whereas the typical behavior is given by the exponential decay

$$\ln \overline{q_L(T_c)} \underset{L \rightarrow \infty}{\propto} -\sqrt{L}. \tag{54}$$

We refer to [32] and references therein for the discussion of similar examples of averaged values dominated by rare events in the field of quantum spin chains.

4.4. Notion of ‘infinite disorder fixed point’

The probability distribution $Q_L(l_a)$ of the attached length l_a in a given sample has the form of a Boltzmann measure

$$Q_L(l_a) = \frac{e^{-U(l_a)}}{\sum_{l_a=0}^L e^{-U(l_a)}} \tag{55}$$

over the random walk potential

$$U(l_a) = -\beta \sum_{i=1}^{l_a} u_i + l_a \ln K = -\sum_{i=1}^{l_a} \ln a_i. \tag{56}$$

The continuum analog of equation (55) is then the Boltzmann measure over a Brownian potential (see equation 10 in [27]). We refer to [27] and references therein for more detailed discussions, but the essential property is that the Boltzmann measure over a Brownian valley is very concentrated, in a region of order $O(1)$, around the absolute minimum of the Brownian potential. This can be seen for instance on the probability distribution of the partition function of equation (38) or equation (84) of [27]. The physical reason is that with respect to the absolute minimum $U_{\min} = U(l^*)$ of the Brownian potential, the potential grows typically as $[U(l) - U(l^*)] \propto \sqrt{|l - l^*|}$, and thus the corresponding Boltzmann weight decays very rapidly as $e^{-(\text{cst})\sqrt{|l - l^*|}}$. It is only in rare configurations that $[U(l) - U(l^*)]$ is finite for large separation $|l - l^*|$, since it occurs with probability $1/|l - l^*|^{3/2}$ (the well-known return probability of a Brownian motion at large distance).

Within our present notations, all this means that in a given disordered sample at a given temperature, the thermal probability distribution $Q_L(l_a)$ is typically concentrated in a finite

region around the point l_a^* where the random potential of equation (56) reaches its minimum on the interval $0 \leq l_a \leq L$

$$Q_L(l_a) \simeq \delta(l_a - l_a^*). \quad (57)$$

This means that in a given sample, the contact density of equation (14) is essentially given by its thermal average

$$\theta_L \simeq \langle \theta_L \rangle = \frac{l_a^*}{L}. \quad (58)$$

4.5. Sample-to-sample fluctuations of the contact density

We now discuss the sample-to-sample fluctuations of l_a^* and of the contact density. The disorder average of the probability distribution $Q_L(l_a)$ can be obtained as

$$\overline{Q_L(l_a)} \simeq \overline{\delta(l_a - l_a^*)} = \pi_L(l_a), \quad (59)$$

where $\pi_L(l_a^*)$ is the probability distribution of the minimum l_a^* over the samples.

Taking into account the two parameters F_0 (equation (25)) and σ (equation (27)) that characterize the large scale properties of the random walk $U(l_a)$, the probability distribution $\pi_L(l_a^*)$ of the minimum l_a^* can be obtained as

$$\pi_L(l_a^*) = \frac{\psi^{(F_0)}(L - l_a^*)\psi^{(-F_0)}(l_a^*)}{\int_0^L dl \psi^{(F_0)}(L - l^*)\psi^{(-F_0)}(l^*)}, \quad (60)$$

where

$$\psi^{(F_0)}(l) = \int_0^{+\infty} dU G^{(F_0)}(U, l) \quad (61)$$

and where $G^{(F_0)}(U, l)$ represents the probability for a biased random walk to go from $(0, 0)$ to (U, l) in the presence of an absorbing wall at $U = 0^-$ (see for instance [32] for very similar calculations):

$$G^{(F_0)}(U, l) = \frac{U}{2\sqrt{\pi}(\sigma l)^{3/2}} e^{-\frac{(U-F_0l)^2}{4\sigma l}}. \quad (62)$$

The critical behavior can be analyzed as follows.

4.5.1. Sample-to-sample fluctuations at criticality. At criticality where $F_0 = 0$, the function ψ of equation (61) reads

$$\psi^{F_0=0}(l) = \int_0^{+\infty} dU \frac{U}{2\sqrt{\pi}(\sigma l)^{3/2}} e^{-\frac{U^2}{4\sigma l}} = \frac{1}{\sqrt{\pi\sigma l}}. \quad (63)$$

The distribution of equation (60) for the position l_a^* then reads

$$\overline{Q_L(l_a^*)} = \pi_L^{(T_c)}(l_a^*) = \frac{1}{\pi\sqrt{l_a^*(L - l_a^*)}}. \quad (64)$$

As a consequence, we find that the contact density $\theta \simeq l_a^*/L$ is distributed at criticality over the samples with the law

$$\mathcal{P}_{T_c}(\theta) = \frac{1}{\pi\sqrt{\theta(1-\theta)}}. \quad (65)$$

4.5.2. Sample-to-sample fluctuations in the delocalized phase. In the delocalized phase where $F_0 < 0$, the position l_a^* of the minimum on $[0, L]$ of the Brownian potential of positive drift remains finite as $L \rightarrow +\infty$, and the probability distribution of equation (60) can be obtained as

$$\pi_L^{(T>T_c)}(l_a^*) \simeq_{L \rightarrow \infty} \frac{\psi^{(-F_0)}(l_a^*)}{\int_1^{+\infty} dl \psi^{(-F_0)}(l)} \quad (66)$$

with

$$\psi^{(-F_0)}(l_a) = \int_0^{+\infty} dU \frac{U}{2\sqrt{\pi}(\sigma l_a)^{3/2}} e^{-\frac{(U+F_0 l_a)^2}{4\sigma l_a}} = \frac{F_0^2(\sigma l_a)^{1/2}}{\sigma^2 2\sqrt{\pi}} \int_0^{+\infty} dz z e^{-l_a \frac{F_0^2}{4\sigma}(1+z)^2}. \quad (67)$$

In particular, it decays exponentially as

$$\overline{Q_L(l_a)} = \pi_L^{(T>T_c)}(l_a) \underset{l_a \rightarrow +\infty}{\propto} e^{-\frac{F_0^2}{4\sigma} l_a}. \quad (68)$$

with the correlation length

$$\xi_a = \frac{4\sigma}{F_0^2} \underset{T \rightarrow T_c^+}{\propto} \frac{1}{(T - T_c)^2}. \quad (69)$$

This should be compared with the typical decay of equation (45). As a consequence, the finite-size scaling properties in the critical region involve the correlation length exponent

$$\nu_{FS} = 2 \quad (70)$$

and not the typical correlation exponent that appears in equation (47).

4.5.3. Sample-to-sample fluctuations in the localized phase. In the localized phase, where $F_0 > 0$, it is the detached length $l_d^* = L - l_a^*$ that remains finite as $L \rightarrow +\infty$, and the disorder-averaged probability of the detached length l_d can be obtained as

$$\overline{P_L(l_d)} = \pi_L^{(T<T_c)}(l_a^* = L - l_d^*) \simeq_{L \rightarrow \infty} \frac{\psi^{(F_0)}(l_d)}{\int_1^{+\infty} dl \psi^{(F_0)}(l)} \quad (71)$$

with

$$\psi^{(F_0)}(l_d) = \int_0^{+\infty} dU \frac{U}{2\sqrt{\pi}(\sigma l_d)^{3/2}} e^{-\frac{(U+F_0 l_d)^2}{4\sigma l_d}}. \quad (72)$$

In particular, it decays exponentially as

$$\overline{P_L(l_d)} \underset{l_d \rightarrow +\infty}{\propto} e^{-\frac{F_0^2}{4\sigma} l_d} \quad (73)$$

with the correlation length

$$\xi_d = \frac{4\sigma}{F_0^2} \underset{T \rightarrow T_c^-}{\propto} \frac{1}{(T_c - T)^2}. \quad (74)$$

This should be compared with the typical decay of equation (49). Again, this means that the finite-size scaling properties in the critical region are governed by the correlation length exponent $\nu_{FS} = 2$ of equation (70).

5. Discussion

5.1. Lack of self-averaging of the order parameter at criticality

Outside criticality, the contact density is self-averaging in the thermodynamic limit $L \rightarrow +\infty$

$$\mathcal{P}_{T < T_c}(\theta) = \delta(\theta - 1) \quad (75)$$

$$\mathcal{P}_{T > T_c}(\theta) = \delta(\theta), \quad (76)$$

but exactly at criticality, we have obtained that the contact density θ remains distributed at criticality over the samples with the law of equation (65):

$$\mathcal{P}_{T_c}(\theta) = \frac{1}{\pi \sqrt{\theta(1-\theta)}}. \quad (77)$$

The lack of self-averaging of densities of extensive thermodynamic observables at random critical points whenever disorder is relevant has been studied in [33–35]. The main idea is that off-criticality, self-averaging is ensured by the finiteness of the correlation function ξ that allows us to divide a big sample $L \gg \xi(T)$ into a large number of nearly independent sub-samples. However, at criticality where the correlation length diverges $\xi = +\infty$, even big samples cannot be divided into nearly independent sub-samples, and one obtains a lack of self-averaging. The cases considered in [33–35] were second-order phase transitions, but the result of equation (65) means that the contact density is not self-averaging at the random first-order transition studied in this paper.

5.2. Physical interpretation of two correlation length exponents

The presence of two different correlation length exponents $\nu_{av} = 2$ and $\nu_{typ} = 1$ whenever the disorder is equivalent to a Brownian potential $U(x)$ is well known in other contexts, in particular for random walks in random media [21, 27], for the quantum transverse field Ising chain [4, 32] and more generally in other models described by the same ‘infinite disorder fixed point’ (see the review [6] and references therein). The physical interpretation is the following [4]:

- (i) The first length scale corresponds to the length ξ_{typ} where the mean value $\overline{[U(L) - U(0)]} = F_0 L$ is of order 1, which yields

$$\xi_{typ} \sim \frac{1}{F_0^{\nu_{typ}}} \quad \text{with} \quad \nu_{typ} = 1. \quad (78)$$

- (ii) The second length scale corresponds to the length ξ_{av} where most of the samples indeed have $(U(L) - U(0)) \sim F_0 L \pm \sqrt{\sigma L}$ of the same sign of the mean value, i.e. the scale $\sqrt{\sigma L}$ of the fluctuations should be of the same order of the mean value, which yields

$$\xi_{av} \sim \frac{1}{F_0^{\nu}} \quad \text{with} \quad \nu_{av} = 2. \quad (79)$$

We have found that the finite-size scaling properties for disorder-averaged values over the samples are governed by $\nu_{FS} = \nu_{av} = 2$ which actually saturates the general bound $\nu_{FS} \geq 2/d_{dis}$ [36], where d_{dis} is the dimensionality of the disorder (here $d_{dis} = 1$), whereas the typical exponent is smaller $\nu_{typ} = 1$. This possibility of a first-order transition that remains first order in the presence of quenched disorder has already been discussed in [36] and in section VII A of [4].

6. Conclusions and perspectives

In this paper, we have studied the random wetting transition on the Cayley tree. We have obtained that the transition, which is first order in the pure case, remains first order in the presence of disorder, but that there exists two diverging length scales in the critical region: the typical correlation length diverges with the exponent $\nu_{\text{typ}} = 1$, whereas the averaged correlation length diverges with the bigger exponent $\nu_{\text{av}} = 2$ which governs the finite-size scaling properties over the samples. We have described the relations with the previously studied models that are governed by the same ‘infinite disorder fixed point’. We have given detailed results on the statistics of the free energy and on the statistics of the contact density $\theta = l_a/L$ which constitutes the order parameter in wetting transitions. In particular, we have obtained that at criticality, the contact density is not self-averaging but remains distributed over the samples in the thermodynamic limit, with the distribution $\mathcal{P}_{T_c}(\theta) = 1/(\pi\sqrt{\theta(1-\theta)})$.

As explained at the beginning, our physical motivation to consider such a model of random wetting on the Cayley tree was to better understand the similarities and differences with two other types of models involving directed polymers and frozen disorder described in the points (i) and (ii) of section 1. The results given in this paper suggest the following conclusions:

- (i) The freezing transition of the directed polymer in a random medium on the Cayley tree [7] is of a very different nature, since it is the exponent $\nu = 2$ which governs the free-energy singular part (instead of $\nu_{\text{typ}} = 1$ here). The appearance of a smaller exponent $\nu' = 1$ in some finite-size properties in the critical region [8, 9] should then be explained with another mechanism.
- (ii) Let us now compare with the wetting and the Poland–Scheraga model of DNA denaturation with a loop exponent $c > 2$ that we have studied numerically in [12, 13]. We find that the random wetting model considered in this paper (that corresponds to the limit of loop exponent $c \rightarrow \infty$ of the model studied in [12]) presents very similar critical behavior: in both cases, the transition remains first order with $\nu_{\text{typ}} = 1$ and a finite contact density, but the finite-size scaling properties over the samples involve the exponent $\nu_{FS} = 2$. We thus hope that the exact results obtained here for the special case $c \rightarrow \infty$ will help to better characterize the transition in the region of finite-loop exponent $2 < c < +\infty$.

Appendix A. Properties of pure wetting transition $u_i = u_0$

In this appendix, we describe the critical properties of the pure wetting transition on the Cayley tree to compare with the disordered case considered in the text.

A.1. Finite-size partition function

When all the energies u_i take the same value u_0 , the ratio R_L of equation (6) simply reads

$$\begin{aligned} R_L^{\text{pure}} &= \left(\frac{e^{\beta u_0}}{K}\right)^L + \frac{(K-1)}{K} \left[1 + \left(\frac{e^{\beta u_0}}{K}\right) + \left(\frac{e^{\beta u_0}}{K}\right)^2 + \dots + \left(\frac{e^{\beta u_0}}{K}\right)^{L-1} \right] \\ &= \left(\frac{e^{\beta u_0}}{K}\right)^L + \frac{(K-1)}{K} \times \frac{\left(\frac{e^{\beta u_0}}{K}\right)^L - 1}{\left(\frac{e^{\beta u_0}}{K} - 1\right)}. \end{aligned} \tag{A.1}$$

The critical temperature T_c then corresponds to the point where the energy per link u_0 exactly compensates for the entropy per link ($\ln K$) of the free walk

$$e^{\beta_c u_0} = K \quad \rightarrow \quad T_c^{\text{pure}} = \frac{u_0}{\ln K}. \tag{A.2}$$

A.2. Delocalized phase $e^{\beta u_0} < K$

In the delocalized phase, the leading term of equation (A.1) remains finite,

$$R_L^{\text{pure}}(T > T_c) \underset{L \rightarrow \infty}{\simeq} \frac{(K-1)}{(K - e^{\beta u_0})} + \dots, \quad (\text{A.3})$$

and the excess free energy per monomer due to the wall in the thermodynamic limit $L \rightarrow +\infty$ (equation (9)) vanishes

$$f_\infty(T > T_c) = 0. \quad (\text{A.4})$$

The finite-size behavior of the full free-energy difference of equation (8) reads near criticality

$$\begin{aligned} F_L^{\text{diff}}(T > T_c) &= F_L(T > T_c) - F_L^{\text{deloc}} = -T \ln R_L \\ &= -T \ln \frac{(K-1)}{(K - e^{\beta u_0})} \underset{T \rightarrow T_c^+}{\simeq} -T_c \ln \frac{1}{T - T_c}. \end{aligned} \quad (\text{A.5})$$

In the delocalized phase, the probability distribution $Q_L(l_a)$ of the attached length l_a of equation (10) remains finite as $L \rightarrow \infty$

$$Q_L(l_a) \equiv P_L(l_d = L - l_a) \underset{L \rightarrow \infty}{\simeq} \left(1 - \frac{e^{\beta u_0}}{K}\right) (K e^{-\beta u_0})^{-l_a} = Q_\infty(l_a). \quad (\text{A.6})$$

It decays exponentially

$$Q_\infty(l_a) \underset{l_a \rightarrow +\infty}{\simeq} e^{-\frac{l_a}{\xi_{\text{deloc}}(T)}}, \quad (\text{A.7})$$

where the correlation length reads

$$\xi_{\text{deloc}}^{\text{pure}}(T > T_c) = \frac{1}{u_0(\beta_c - \beta)} \underset{T \rightarrow T_c^+}{\propto} \frac{1}{(T - T_c)^{\nu_{\text{pure}}}} \quad \text{with} \quad \nu_{\text{pure}} = 1. \quad (\text{A.8})$$

The thermally averaged contact density $\langle \theta_L \rangle$ of equation (15) vanishes as

$$\langle \theta_L \rangle \underset{L \rightarrow \infty}{\simeq} \frac{1}{L} \sum_{l_a=0}^{\infty} l_a Q_\infty(l_a) = \frac{1}{L(K e^{-\beta u_0} - 1)}, \quad (\text{A.9})$$

i.e. in the critical region, it behaves as

$$\langle \theta_L \rangle \underset{L \rightarrow \infty}{\simeq} = \frac{1}{L(\beta_c - \beta)}. \quad (\text{A.10})$$

A.3. Localized phase $e^{\beta u_0} > K$

In the localized phase $T < T_c^{\text{pure}}$, the leading term of the partition function of equation (A.1) is exponentially large in L

$$R_L^{\text{pure}}(T < T_c) \underset{L \rightarrow \infty}{\simeq} \left(\frac{e^{\beta u_0}}{K}\right)^L \left[1 + \frac{(K-1)}{(e^{\beta u_0} - K)}\right] + O(1). \quad (\text{A.11})$$

The excess free energy per monomer due to the wall in the thermodynamic limit $L \rightarrow +\infty$ (equation (9)) reads

$$f_\infty(T < T_c) = -(u_0 - T \ln K) = -(T_c^{\text{pure}} - T) \ln K, \quad (\text{A.12})$$

i.e. it vanishes linearly as the temperature T approaches the critical value T_c^{pure} . The transition is thus first order.

The internal energy of equation (16) becomes from equation (A.11)

$$E_L(T < T_c) \underset{L \rightarrow +\infty}{\simeq} -Lu_0 + \frac{(K-1)u_0 e^{\beta u_0}}{(e^{\beta u_0} - K)(e^{\beta u_0} - 1)} + \dots \quad (\text{A.13})$$

The intensive energy of equation (19) remains constant in the whole low-temperature phase

$$e_\infty(T < T_c) = -u_0 \quad (\text{A.14})$$

and presents a jump at criticality.

The detached length l_d remains a finite random variable in the thermodynamic limit $L \rightarrow \infty$ with the following distribution (equation (13)):

$$P_\infty(l_d) = \frac{\delta_{l_d=0} + \theta(l_d \geq 1) \frac{K-1}{K} \left(\frac{e^{\beta u_0}}{K}\right)^{-l_d}}{\left[1 + \frac{(K-1)}{K \left(\frac{e^{\beta u_0}}{K} - 1\right)}\right]}. \quad (\text{A.15})$$

It decays exponentially

$$P_\infty(l_d) \underset{l_d \rightarrow +\infty}{\simeq} e^{-\frac{l_d}{\xi(T)}}, \quad (\text{A.16})$$

where the correlation length diverges as

$$\xi^{\text{pure}}(T < T_c) = \frac{1}{u_0(\beta - \beta_c)} \underset{T \rightarrow T_c^-}{\propto} \frac{1}{(T_c - T)^{\nu_{\text{pure}}}} \quad \text{with} \quad \nu_{\text{pure}} = 1. \quad (\text{A.17})$$

A.4. Critical point $e^{\beta_c u_0} = K$

Exactly at criticality, the ratio of equation (A.1) reads

$$R_L^{\text{pure}}(T_c) = 1 + \frac{(K-1)}{K} L \quad (\text{A.18})$$

so that the free-energy difference is logarithmic in L

$$\begin{aligned} F_L^{\text{diff}}(T_c) &= F_L^{\text{pure}}(T_c) - F_L^{\text{deloc}}(T_c) = -T_c \ln R_L \\ &= -T_c \ln \left[1 + \frac{(K-1)}{K} L\right] \underset{L \rightarrow \infty}{\simeq} -T_c \ln L. \end{aligned} \quad (\text{A.19})$$

The recurrence of equation (17) for the energy becomes

$$E_{L+1} = \frac{K R_L}{K R_L + (K-1)} (-u_0 + E_L) = \frac{K + (K-1)L}{K + (K-1)(L+1)} (u_0 + E_L) \quad (\text{A.20})$$

leading to

$$E_L^{\text{pure}}(T_c) = -u_0 \frac{KL + (K-1)\frac{L(L-1)}{2}}{K + (K-1)L} = -u_0 \frac{(L-1)}{2} \times \frac{1 + \frac{2K}{(K-1)(L-1)}}{1 + \frac{K}{(K-1)L}} \quad (\text{A.21})$$

so that the energy per monomer in the thermodynamic limit $L \rightarrow \infty$ is

$$e_\infty^{\text{pure}}(T_c) = -\frac{u_0}{2}, \quad (\text{A.22})$$

i.e. exactly in the middle of the jump between $e_\infty^{\text{pure}}(T < T_c) = -u_0$ (equation (A.14)) and $e_\infty^{\text{pure}}(T > T_c) = 0$.

The probability distribution $Q_L(l_a)$ of the attached length l_a for a system of size L becomes at criticality

$$Q_L(l_a) = \frac{\left[\delta_{l_a=L} + \frac{(K-1)}{K} \theta(l_a < L)\right]}{1 + \frac{(K-1)}{K} L}, \quad (\text{A.23})$$

i.e. it is essentially flat over all values $0 \leq l_a \leq L$. The two phases coexist with proportions $(l_a, l_d = L - l_a)$. So the value of equation (A.21) simply means that the domain wall between the two phases is on average at the middle ($l_a = l_d = L/2$), but the domain wall is actually anywhere on the interval $0 \leq l_a \leq L$ with a flat distribution.

Appendix B. Disordered case: transition temperatures T_n for the averaged moments $\overline{Z_L^n}$ of the partition function

Since in other disordered models, one sometimes considers the series T_n of critical temperatures for the moments $\overline{Z_L^n}$ of the partition function, we discuss their properties in this appendix. For $n = 1$, one sees that the annealed partition function satisfies the recursion (see equation (7))

$$\overline{R_{L+1}} = \frac{\overline{e^{\beta u}}}{K} \overline{R_L} + \frac{(K-1)}{K}. \quad (\text{B.1})$$

This is equivalent to the pure case recursion with the change $e^{\beta u_0} \rightarrow \overline{e^{\beta u}}$ so that the annealed critical temperature is determined by the condition (see equation (A.2))

$$\overline{e^{\beta u}} = K, \quad (\text{B.2})$$

i.e. in terms of the exponent $\mu(T)$ introduced in equation (30), this corresponds to the condition

$$\mu(T_1) = 1. \quad (\text{B.3})$$

More generally from the power law of equation (29), it is clear that the transition temperature T_n for the moments $\overline{Z_L^n}$ of order n is determined by the condition

$$\mu(T_n) = n. \quad (\text{B.4})$$

For instance, for the case of the Gaussian distribution of equation (3), the expression of equation (34) yields

$$T_n = T_c \frac{1 + \sqrt{1 + 2n \frac{\Delta^2}{T_c^2 \ln K}}}{2}. \quad (\text{B.5})$$

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